Weak Solutions of
Partial Differential Equations

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Abstract

This paper is an introduction to weak solutions of partial differential equations (PDEs), an essential tool for any mathematician working in the field of PDEs. Our aim is to motivate and present the basic theory of weak solutions of linear elliptic operators. The prerequisites are multivariable calculus, measure theory, and basic functional analysis. Preferably the reader has some acquaintance with PDEs. As such this paper should be accessible to most beginning graduate students.

1 Introduction

Partial differential equations (PDEs) play an important role in many scientific disciplines outside mathematics. Most physical laws are described by partial differential equations. A classical example is **Poisson's equation**, which arises when computing electrical potentials, defined on a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ as

$$\Delta u(x) := \sum_{j=1}^{n} \frac{\partial^2 u(x)}{\partial x_j^2} = 0, \quad x \in \Omega.$$

Many PDEs of interest can be very difficult to solve. In fact there are many equations which it is unclear whether they admit solutions. One approach to prove the existence of solutions is based on the idea to consider equations in a so called weak sense. This led to the concept of weak solutions which we will introduce in this paper.

In section 2 we briefly discuss classical solutions of PDEs and their limitations. Section 3 gives a gentle introduction to Sobolev spaces, which are necessary to formulate the definition of weak solutions. Finally in section 4 we introduce weak solutions and discuss their usefulness.

In this text $\Omega \subset \mathbb{R}^n$ will denote an open bounded set and we consider functions with domain $\overline{\Omega}$ and image in $\mathbb{R}$. For notational convenience we use the notation ∂_j instead of $\frac{\partial}{\partial x_j}$ to denote the ordinary partial derivative given by the difference quotient

$$\partial_j u(x) = \frac{\partial u(x)}{\partial x_j}(x) = \lim_{h \to 0} \frac{u(x + h\mathbf{e}_j) - u(x)}{h}.$$

We will consider a general class of equations on the form

$$\begin{cases} Lu(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where the operator, L is given by

$$Lu = -\sum_{i,j=1}^{n} \partial_j \big(a^{ij}(x)\partial_i u(x)\big) + \sum_{i=1}^{n} b^i(x)\partial_i u(x) + c(x)u(x). \tag{2}$$

The main result is presented in section four which says the there exists solutions to equation (1) in the weak sense if we impose some restrictions on the differential operator L.

2 Classical PDE Theory

A PDE is said to be **well posed** if:

1. It exist a solution to the equation.
2. The solution unique.
3. The solution is continuous.

A solution which is unique and C^k differentiable where k is the order the equation is said to be a **classical solution**. E.g. a unique C^2 solution to Poisson's equation would be a classical solution. We will not be concerned with classical solutions in this text, however they are studied in detail in many textbooks. Two books which treat much of the classical theory are [Eva10] and [Fol95].

As we mentioned there are many equations for which it is impossible to prove the existence of classical solutions. The domain and boundary conditions play a large role in proving existence of solutions. The idea behind weak solutions is to consider a larger class of functions as possible solutions than the standard classical solutions (which are at least C^k continuous). The proper spaces to search for these solutions are the so called **Sobolev spaces**, $W^{k,p}(\Omega)$, which we introduce in the next section.

Consider a differential operator on the form presented in equation (2). We will assume that the coefficients functions a^{ij}, b^i, c are members of $C^2(\Omega)$ and that $a^{ij} = a^{ji}$. We define

$$\mathbf{A}(\mathbf{x}) = [a^{ij}(x)]_{i,j}, \quad \mathbf{b}(x) = (b^1(x), \dots, b^n(x)).$$

Then $\mathbf{A}$ is a symmetric matrix and we can write equation (2) as

$$Lu = -\nabla \cdot \mathbf{A}(x)\nabla u(x) + \mathbf{b}(x) \cdot \nabla u(x) + c(x)u(x),$$

where $\nabla := (\partial_1, \dots, \partial_n)$.

Definition 2.1. A partial differential operator, L, on the form

$$Lu = -\sum_{i,j=1}^{n} \partial_j \big(a^{ij}(x)\partial_i u(x)\big) + \sum_{i=1}^{n} b^i(x)\partial_i u(x) + c(x)u(x),$$

is said to be *uniformly elliptic* if there exists a positive constant η such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \eta|\boldsymbol{\xi}|^2,$$

for a.e. $x \in \Omega$ and every $\boldsymbol{\xi} \in \mathbb{R}^n$.

The criterion for uniform ellipticity is equivalent to the matrix $\mathbf{A}(x)$ being positive definite with all eigenvalues satisfying $\lambda_i \geq \eta$ for a.e. $x \in \Omega$. A simple example of an elliptic operator is the Laplacian given by

$$-\Delta u(x) = \sum_{j=1}^{n} \partial_j^2 u(x).$$

The corresponding coefficients are

$$\mathbf{A}(x) = [\delta_{ij}], \quad \mathbf{b} = 0, \quad c = 0.$$

3 Sobolev Spaces

Sobolev spaces are subspaces of L^p spaces which are suitable spaces to look for solutions of differential equations. We will just scratch on their surface in this text but they are well studied. A very thorough reference is [AF03], and most advanced textbooks on PDEs such as [GT15], [Eva10] treat them in detail.

We start by introducing multiindex notation which simplify the notation of mixed partial derivatives.

Definition 3.1. We say that $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ is a n-dimensional *multiindex* of order $|\alpha|$, where

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Multiindices can be used to represent differential operators. Given a multi-index α we define,

$$\partial^\alpha u(x) := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} u(x).$$

The above shorthand can simplify expressions involving many differentials significantly. Recall that the space $L^p(\Omega)$ consists of all measurable functions which satisfy

$$||f||_p = \int_\Omega |f|^2\, dx < \infty$$

The space of **locally** p-integrable functions is defined as

$$L^p_{\text{loc}}(\Omega) := \{f \ : f \in L^p(K), \text{ for every compact } K \subset \Omega\}.$$

Definition 3.2. Given two functions $u, v \in L^1_{\text{loc}}(\Omega)$ and a multiindex α we say that v the α-th weak derivative of u, written $D^\alpha u = v$, if

$$(-1)^{|\alpha|} \int_\Omega u\, \partial^\alpha \varphi \, dx = \int_\Omega v\, \varphi \, dx, \quad \forall \varphi \in C_c^\infty(\Omega).$$

By convention we define $D^\alpha u = u$ when $\alpha = (0, \ldots, 0)$. We are going to denote the weak derivatives of first order by $D_j u$. I.e. $D_j u$ is the weak derivative corresponding to the multiindex with zeros in every entry except for the j-th position where it is 1. The weak derivatives can be thought of as an extension of classical derivatives. If $u \in C^1(\overline{\Omega})$ then an integration by parts shows that

$$-\int_\Omega u \partial_j \varphi \, dx = \int_\Omega \partial_j u\, \varphi \, dx, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Thus $\partial_j u = D_j u$, i.e. they belong to the same function class in $L^p(\Omega)$. Which means that if the derivative exists in the classical sense it is also the weak derivative of u. We are now ready to define the Sobolev spaces.

Definition 3.3. For $k \in \mathbb{N}$ and $p \geq 1$ we define the *Sobolev space* $W^{k,p}(\Omega) \subset L^p(\Omega)$ as the space of functions for which all weak derivatives up to order k exists and are in $L^p(\Omega)$.

It is easy to show that weak derivatives satisfies some properties we recognise from classical derivatives, such as linearity and a chain rule. We state that result below as a lemma.

Lemma 3.1 (Properties of weak derivatives)**.** *Let $u, v \in W^{k,p}(\Omega)$, $f \in C^k(\Omega)$ and $\lambda, \nu \in \mathbb{R}$, then*

$$\begin{aligned} D_j(\lambda u + \nu v) &= \lambda D_j u + \nu D_j v, \\ D_j(fu) &= D_j(f)\, u + D_j(u)\, f. \end{aligned}$$

Sobolev spaces are linear spaces and the space $W^{k,p}(\Omega)$ can be equipped with the norm

$$||u||_{k,p} = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \, dx \right)^{\frac{1}{p}}.$$

This norm makes $W^{k,p}(\Omega)$ complete, i.e. a Banach space. The closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$ is an important subspace denoted by $W_0^{k,p}(\Omega)$. When $k = 1$ and $p = 2$, then $W_0^{1,2}(\Omega)$ is Hilbert space (is often denoted as H_0^1). The Hilbert space structure of $W_0^{1,2}$ is used to prove the existence of weak solutions for second order linear elliptic operators.

4 Weak solutions

In this section we define weak solutions and state one of the existence theorems for elliptic operators. Let us restate the equation from the introduction

$$\begin{cases} Lu = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

where the operator, L is given by

$$Lu = -\sum_{i,j=1}^{n} \partial_j \big(a^{ij} \partial_i u\big) + \sum_{i=1}^{n} b^i \partial_i u + cu. \tag{4}$$

Let $\varphi \in C_c^{\infty}(\Omega)$ and assume that $u \in C^2(\Omega)$ is a classical solution of equation (3). Then we may multiply both sides of equation (3) by φ, and integrate which yields identity

$$\int_{\Omega} \Big(-\sum_{i,j=1}^{n} \partial_j \big(a^{ij} \partial_i u\big) + \sum_{i=1}^{n} b^i \partial_i u + cu \Big) \varphi \, dx = \int_{\Omega} f \varphi \, dx.$$

An integration by parts applied to the first term of the left hand side integral yields the identity

$$\int_{\Omega} \sum_{i,j=1}^{n} a^{ij} \partial_j \varphi \partial_i u + \Big(\sum_{i=1}^{n} b^i \partial_i u + cu \Big) \varphi \, dx = \int_{\Omega} f \varphi \, dx. \tag{5}$$

Note that even though the the operator L is a second order operator, equation (5) only involves first order derivatives. Equation (5) also makes sense for elements of $W_0^{1,2}(\Omega)$ if we replace the classical derivatives with weak derivatives. This induces a bilinear form $\mathcal{L} : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$, defined by

$$\mathcal{L}(u, \varphi) := \int_{\Omega} \sum_{i,j=1}^{n} a^{ij} \partial_j \varphi \partial_i u + \Big(\sum_{i=1}^{n} b^i \partial_i u + cu \Big) \varphi \, dx.$$

The definition of weak derivatives is based on this idea.

Definition 4.1. An element $u \in W_0^{1,2}(\Omega)$ is said to be a *weak solution* of equation (4) if

$$\mathcal{L}(u, \varphi) = \int_{\Omega} f \varphi \, dx, \quad \forall \, \varphi \in W_0^{1,2}(\Omega). \tag{6}$$

The power of the definition of weak solutions is that $W_0^{1,2}$ is a Hilbert space, which shows that equation (6) is a Hilbert space equation. Hence tools from functional analysis can be applied to solve the equation in the weak sense. An application of the Fredholm Alternative and Lax-Milgram's Theorem combined with several technical estimates yields the following theorem.

Theorem 4.1. *Let L be a uniformly elliptic partial differential operator on the form*

$$Lu = -\sum_{i,j=1}^{n} \partial_j \big(a^{ij} \partial_i u\big) + \sum_{i=1}^{n} b^i \partial_i u + cu,$$

with coefficients a^{ij}, b^i, c in $L^{\infty}(\Omega)$. Then one of the following holds.

For each $f \in L^2(\Omega)$ *there exists a unique solution to the problem*

$$Lu = f, \text{ in } \Omega, \quad u = 0, \text{ on } \partial\Omega.$$

or there exists a weak solution $u \neq 0$ *of the homogeneous equation*

$$Lu = 0, \text{ in } \Omega, \quad u = 0, \text{ on } \partial\Omega.$$

Theorem 4.1 can be used to prove the existence of solutions to many elliptic PDEs in the weak sense. However existence is only one part of the theory of weak solutions. After proving the existence of solutions it can often be shown that the solution is smoother than $W_0^{1,2}$. This is known as regularity theory and regularity of solutions.

Two of 20th century most famous mathematicians John Nash and Ennio De Giorgi contributed to the regularity theory of elliptic operators and proved that with a sufficiently smooth boundary and right hand side in equation (3) the solutions are Hölder continuous [Nas58], [De 57]. Moser found a new proof of this result inspired by De Giorgi [Mos60]. Their techniques has been applied to other classes of PDEs and are often referred to as De Giorgi-Nash-Moser techniques.

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