Large Deviations and Weak Convergence of Measures

with applications to Monte Carlo Estimators

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Motivation

- Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and real valued random variable $X : \Omega \to \mathbb{R}$, with distribution $\mu := \mathbb{P} \circ X^{-1}$.

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- In many applications we are interested in computing the expected value $\mathbb{E}[X]$

$$\theta = \mathbb{E}[X] = \int_{\Omega} X \,\mathrm{d}\mathbb{P}\,.$$

- In practice, it may not be possible to compute this integral and Monte Carlo (MC) methods are often used to simulate θ.

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- *Main idea behind MC-method:* Strong law of large numbers (SLLN) implies that $\theta_n(\omega) \to \theta$ almost surely.

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$$\mathbb{V}[\mathbf{1}_A] = p(1-p).$$

- For rare events the variance and expectation are almost the same!

$$\frac{\mathbb{V}[\mathbf{1}_A]}{\mathbb{E}[\mathbf{1}_A]} = 1 - p, \quad n \gtrsim \frac{z_{1-\alpha/2}^2}{\varepsilon^2 p}$$

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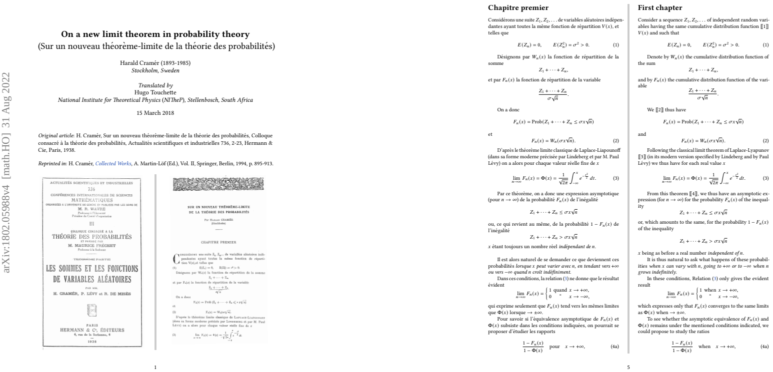
- Cramér's Motivation was insurance mathematics and ruin probabilites.
- S.R.S. Varadhan introduced the modern mathematical theory of large deviations. Seminal paper: [5] (Abel Prize for his contributions).

Large Deviations



arXiv:1802.05988v4 [math.HO] 31 Aug 2022

On a new limit theorem in probability theory
(Sur un nouveau théorème-limite de la théorie des probabilités)

Harald Cramér (1893-1985)
Stockholm, Sweden

Translated by
Hugo Touchette
National Institute for Theoretical Physics (NITheP), Stellenbosch, South Africa

15 March 2018

Original article: H. Cramér, Sur un nouveau théorème-limite de la théorie des probabilités, Colloque consacré à la théorie des probabilités, Actualités scientifiques et industrielles 736, 2-23, Hermann & Cie, Paris, 1938.

Reprinted in: H. Cramér, Collected Works, A. Martin-Löf (Ed.), Vol. II, Springer, Berlin, 1994, p. 895-913.

ACTUALITÉS SCIENTIFIQUES ET INDUSTRIELLES
736
CONFÉRENCES INTERNATIONALES DE SCIENCES MATHÉMATIQUES
III
THÉORIE DES PROBABILITÉS
LES SOMMES ET LES FONCTIONS DE VARIABLES ALÉATOIRES
H. CRAMÉR, P. LÉVY et R. DE MISES
PARIS
HERMANN & Cie, ÉDITEURS
1938

SUR UN NOUVEAU THÉORÈME-LIMITE DE LA THÉORIE DES PROBABILITÉS
Par Harald CRAMÉR (Stockholm)
CHAPITRE PREMIER

1

Chapitre premier

Considérons une suite $Z_1, Z_2, \ldots$ de variables aléatoires indépendantes ayant toutes la même fonction de répartition $V(x)$, et telles que

$$E(Z_n) = 0, \qquad E(Z_n^2) = \sigma^2 > 0. \tag{1}$$

Désignons par $W_n(x)$ la fonction de répartition de la somme

$$Z_1 + \cdots + Z_n,$$

et par $F_n(x)$ la fonction de répartition de la variable

$$\frac{Z_1 + \cdots + Z_n}{\sigma\sqrt{n}}.$$

On a donc

$$F_n(x) = \text{Prob}(Z_1 + \cdots + Z_n \leq \sigma x\sqrt{n})$$

et

$$F_n(x) = W_n(\sigma x\sqrt{n}). \tag{2}$$

D'après le théorème limite classique de Laplace-Liapounoff (dans sa forme moderne précisée par Lindeberg et par M. Paul Lévy) on a alors pour chaque valeur réelle fixe de x

$$\lim_{n\to\infty} F_n(x) = \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt. \tag{3}$$

Par ce théorème, on a donc une expression asymptotique (pour $n \to \infty$) de la probabilité $F_n(x)$ de l'inégalité

$$Z_1 + \cdots + Z_n \leq \sigma x\sqrt{n}$$

ou, ce qui revient au même, de la probabilité $1 - F_n(x)$ de l'inégalité

$$Z_1 + \cdots + Z_n > \sigma x\sqrt{n}$$

x étant toujours un nombre réel *indépendant de n*.

Il est alors naturel de se demander ce que deviennent ces probabilités *lorsque x peut varier avec n, en tendant vers $+\infty$ ou vers $-\infty$ quand n croît indéfiniment.*

Dans ces conditions, la relation (3) ne donne que le résultat évident

$$\lim_{n\to\infty} F_n(x) = \begin{cases} 1 & \text{quand } x \to +\infty, \\ 0 & \text{''} \quad x \to -\infty, \end{cases}$$

qui exprime seulement que $F_n(x)$ tend vers les mêmes limites que $\Phi(x)$ lorsque $\to \pm\infty$.

Pour savoir si l'équivalence asymptotique de $F_n(x)$ et $\Phi(x)$ subsiste dans les conditions indiquées, on pourrait se proposer d'étudier les rapports

$$\frac{1 - F_n(x)}{1 - \Phi(x)} \quad \text{pour} \quad x \to +\infty, \tag{4a}$$

First chapter

Consider a sequence $Z_1, Z_2, \ldots$ of independent random variables having the same cumulative distribution function [[1]] $V(x)$ and such that

$$E(Z_n) = 0, \qquad E(Z_n^2) = \sigma^2 > 0. \tag{1}$$

Denote by $W_n(x)$ the cumulative distribution function of the sum

$$Z_1 + \cdots + Z_n,$$

and by $F_n(x)$ the cumulative distribution function of the variable

$$\frac{Z_1 + \cdots + Z_n}{\sigma\sqrt{n}}.$$

We [[2]] thus have

$$F_n(x) = \text{Prob}(Z_1 + \cdots + Z_n \leq \sigma x\sqrt{n})$$

and

$$F_n(x) = W_n(\sigma x\sqrt{n}). \tag{2}$$

Following the classical limit theorem of Laplace-Lyapunov [[3]] (in its modern version specified by Lindeberg and by Paul Lévy) we thus have for each real value x

$$\lim_{n\to\infty} F_n(x) = \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt. \tag{3}$$

From this theorem [[4]], we thus have an asymptotic expression (for $n \to \infty$) for the probability $F_n(x)$ of the inequality

$$Z_1 + \cdots + Z_n \leq \sigma x\sqrt{n}$$

or, which amounts to the same, for the probability $1 - F_n(x)$ of the inequality

$$Z_1 + \cdots + Z_n > \sigma x\sqrt{n}$$

x being as before a real number *independent of n*.

It is thus natural to ask what happens of these probabilities *when x can vary with n, going to $+\infty$ or to $-\infty$ when n grows indefinitely.*

In these conditions, Relation (3) only gives the evident result

$$\lim_{n\to\infty} F_n(x) = \begin{cases} 1 & \text{when } x \to +\infty, \\ 0 & \text{''} \quad x \to -\infty, \end{cases}$$

which expresses only that $F_n(x)$ converges to the same limits as $\Phi(x)$ when $\to \pm\infty$.

To see whether the asymptotic equivalence of $F_n(x)$ and $\Phi(x)$ remains under the mentioned conditions indicated, we could propose to study the ratios

$$\frac{1 - F_n(x)}{1 - \Phi(x)} \quad \text{when} \quad x \to +\infty, \tag{4a}$$

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Figure: Translation of Cramér's publication from French to English by Hugo Touchette [2]

Large Deviations

- A sequence (X_n) of i.i.d. random variables taking values in a Hausdorff topological space $\mathcal{X}$.
- In the theory of large deviations we want to find a rate function $I : \mathcal{X} \to [0, \infty]$, such that

$$\lim_{n\to\infty} \mathbb{P}(X_n \in A) \approx e^{-n \inf_{x\in A} I(x)}.$$

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- Can be used to analyze the convergence of MC estimators (substitute $X_n = \theta_n$).

Empirical Distributions

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$$\delta : \Omega \to \mathbf{M_1}(\mathcal{X}), \quad \delta_{X(\omega)}(A) := \begin{cases} 1, & X(\omega) \in A, \\ 0, & X(\omega) \notin A, \end{cases}$$

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- Does $\mathbf{L}_n$ converge to the distribution μ of $\mathcal{X}$ in $\mathbf{M}_1(\mathcal{X})$?

Topologies on $\mathbf{M}(\mathcal{X})$

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Three important spaces of measures:

1. $\mathbf{M}(\mathcal{X})$ finite signed measures on $\mathcal{X}$ (is a linear space)
2. $\mathbf{M}_+(\mathcal{X})$ nonnegative finite measures on $\mathcal{X}$
3. $\mathbf{M}_1(\mathcal{X})$ probability measures on $\mathcal{X}$

The τ-topology

- Let $f : \mathcal{X} \to \mathbb{R}$ be a bounded measurable function and $\mu \in \mathbf{M}(\mathcal{X})$. Then

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- The maps $\langle f, \cdot \rangle : \mathbf{M}(\mathcal{X}) \to \mathbb{R}$ generate a weak topology on $\mathbf{M}(\mathcal{X})$: the τ-topology
- μ_α converges to μ in the τ-topology iff
$$\lim_{\alpha} \int_{\mathcal{X}} f \, \mathrm{d}\mu_\alpha = \int_{\mathcal{X}} f \, \mathrm{d}\mu \, ,$$
for every bounded measurable function f.

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Definition

Let $\mathcal{X}$ be a metrizable space, then a net (μ_α) in $\mathbf{M}(\mathcal{X})$ *converges weakly* to $\mu \in \mathbf{M}(\mathcal{X})$ if

$$\lim_\alpha \int_\mathcal{X} f \, \mathrm{d}\mu_\alpha = \int_\mathcal{X} f \, \mathrm{d}\mu \,, \quad \text{for every } f \in C_b(\mathcal{X}).$$

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The topology of weak convergence is weaker (has less open sets) than the τ-topology!

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- In Ch. 3 of the thesis we study the the τ-topology and the topology of weak convergence.
- Extend many results from $\mathbf{M_1}(\mathcal{X})$ to $\mathbf{M}_+(\mathcal{X})$.
- Builds upon the work of Varadarajan in [4].

Convergence of MC-estimators

Varadarajan proved in 1958 the empirical distributions of the CMC-estimator converges weakly [3].

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Theorem

Let $\mathcal{X}$ be a separable metrizable space and (X_i) a sequence of i.i.d. random variables taking values in $\mathcal{X}$ with law μ. Then the empirical distributions $\mathbf{L}_n$ converge weakly to μ almost surely, i.e.

$$\mathbb{P}(\{\omega \in \Omega \ : \ \mathbf{L}_n(\omega) \implies \mu\} = 1.$$

We extend this result to the empirical distribution of the importance sampling estimator.

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A sequence of probability measures (μ_n) the *large deviation principle (LDP)* with rate function I if

$$-\inf_U I \leq \liminf_{n\to 0} \frac{1}{n} \log\left[\mu_n(U)\right]$$

for every open set U, and

$$-\inf_C I \geq \limsup_{n\to 0} \frac{1}{n} \log\left[\mu_n(C)\right]$$

for every closed set C.

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3. Identification of the rate function.

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- (μ_n) satisfies *a weak large deviation principle* with rate function I if it satisfies the lower bound and

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for every compact set $K \subset \mathcal{X}$.

- Easier to prove a weak LDP. Can go from weak to full LDP by proving that the sequence (μ_n) is exponentially tight.

Large Deviations of empirical means

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- Consider (X_i) i.i.d. random variables with distribution μ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in the topological linear space $\mathcal{X}$.
- The empirical means $\mathbf{S}_n : \Omega \to \mathcal{X}$, defined by

$$\mathbf{S}_n(\omega) = \frac{1}{n} \sum_{i=1}^{n} X_i(\omega).$$

- The distributions are given by $\mu_n = \mathbb{P} \circ \mathbf{S}_n{}^{-1}$.

Cramér's Theorem

Theorem (Weak Cramér's Theorem)

The sequence (μ_n) of distributions of the empirical means satisfy a weak large deviation principle with a convex rate function $I = \Lambda^*$, and

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_n(A) = -\inf_{x\in A} \Lambda^*(x),$$

for every convex and open $A \subset \mathcal{X}$.

Here Λ^* is the Legendre-Fenchel transform of

$$\Lambda(\lambda) = \Lambda_\mu := (\lambda) = \log \mathbb{E}\left[e^{\langle \lambda, X\rangle}\right] = \log \left[\int_{\mathcal{X}} e^{\langle \lambda, d\rangle}\, \mathrm{d}\mu(x)\right].$$

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$$\mathbb{P}\Big(|\theta_n - \theta| < \varepsilon|\theta| \Big) > 1 - \alpha.$$

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- For large n we can interpret this as

$$\mu_n(A_\varepsilon) \lessapprox e^{-nI(A_\varepsilon)}.$$

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- Rearranging yields:

$$n \gtrapprox \frac{\log(\alpha)}{I(A_\varepsilon)}.$$

Example with $X \sim N(\theta, \sigma^2)$

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- Compare with introduction:

$$n \gtrapprox \frac{z_{1-\alpha/2}^2\sigma^2}{\varepsilon^2\theta^2}.$$

Convexity and Large Deviations

Definition

Let $f : \mathcal{X} \to [-\infty, \infty]$, then the *Legendre-Fenchel transform*of f is the function $f : \mathcal{X}^* \to [-\infty, \infty]$ defined by

$$\begin{aligned} f^*(\lambda) &= \sup\{\langle \lambda, x \rangle - f(x) \ : \ x \in \mathcal{X}\} \\ &= -\inf\{f(x) - \langle \lambda, x \rangle \ : \ x \in \mathcal{X}\}. \end{aligned}$$

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Theorem (Biconjugate Theorem)

Let $f : \mathcal{X} \to (-\infty, \infty]$ not be identically ∞, then $f = f^{**}$ if and only if f is convex and lower semicontinuous.

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Questions

Feel free to ask any questions!

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