Gamma Random Number Generation on GPUs using CUDA

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Introduction

Motivation

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  3. weather forecasting and climate modeling
- Monte Carlo Simulations has been used since the Manhattan Project.
- Require that we can simulate random variables efficiently!

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- The computer architecture of GPUs differ from that of classical central processing units (CPUs).
- Algorithms that perform well on CPUs may not perform well on GPUs and vice versa.
- Challenges when implementing code that require random numbers on GPUs:
  1. Poor library support for complex distributions (e.g. gamma)
  2. Much of the existing literature is focused on CPUs and not GPUs.

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- We describe the implementation and design of efficient random number generation kernels on GPUs.
- Our results show that a 1000× speedup can be achieved when generating gamma random numbers on a consumer grade GPU compared to on a CPU (single thread).

Background

Gamma Distribution

Let $\alpha, \beta > 0$ be real numbers, then the *gamma distribution* $\Gamma(\alpha, \beta)$ has p.d.f.

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

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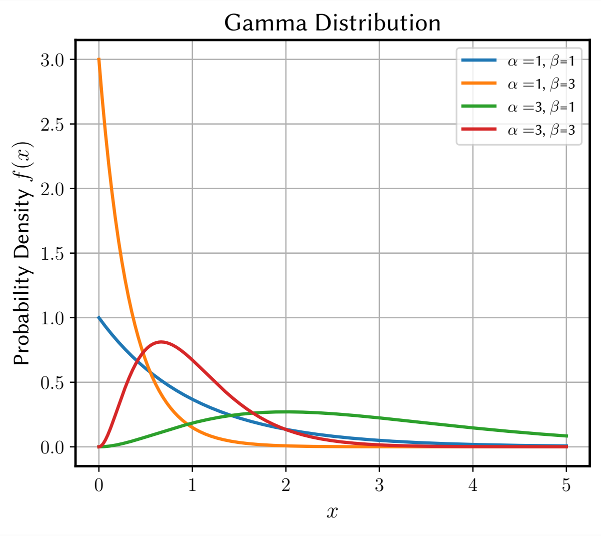

Figure: Gamma Distribution for different shape and scale parameters.

Gamma Distribution

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$$cX_1 + \cdots + cX_n \in \Gamma(\alpha_1 + \cdots + \alpha_n, c\beta).$$
3. If $Y \sim \Gamma(\alpha + 1, 1)$ for some $\alpha > 0$ and $U \sim U(0, 1)$ be independent random variables, then
$$X = YU^{1/\alpha} \sim \Gamma(\alpha, 1).$$

Item 2 and 3 above implies that $\Gamma(\alpha, 1)$ $(\alpha > 1)$ variates can be cheaply transformed to $\Gamma(\alpha, \beta)$ variates for arbitrary α, β. Hence, our focus is on $\Gamma(\alpha, 1)$ generators for $\alpha > 1$.

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- Non-uniform RNGs use uniform RNGs for randomness combined with mathematical transforms to generate samples from other distributions.
- Only one method is used for gamma generation: rejection sampling.

GPU Architecture

- At the highest level a GPU consists of several *streaming multiprocessors (SMs)*.
- The SMs have a *single instruction, multiple threads (SIMT)* design: many compute cores each have their own registers and but are collected in groups which share the same instruction control unit.

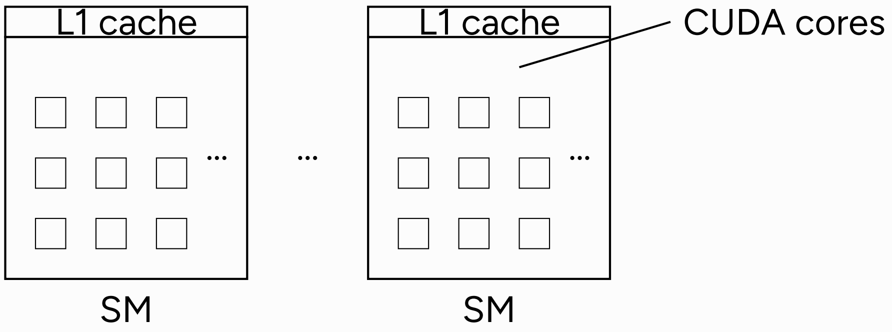

Figure: Illustration of GPU architecture showing SMs, CUDA cores, and L1 cache.

Rejection Sampling

Algorithm 1: Rejection Sampling

Data: Desired distribution f, proposal distribution g, constant M

Result: Sample from distribution X

Input: Initialize $accepted \leftarrow$ false

```
while not accept do
    Sample y ~ Y
    Sample u ~ U(0, 1)
    if u < f(y) / (M·g(y)) then
        accept ← true
    end
    return y
end
```

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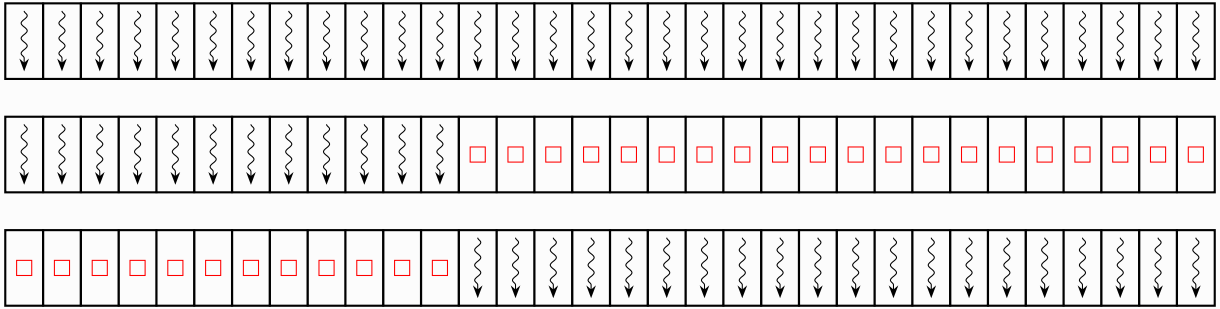


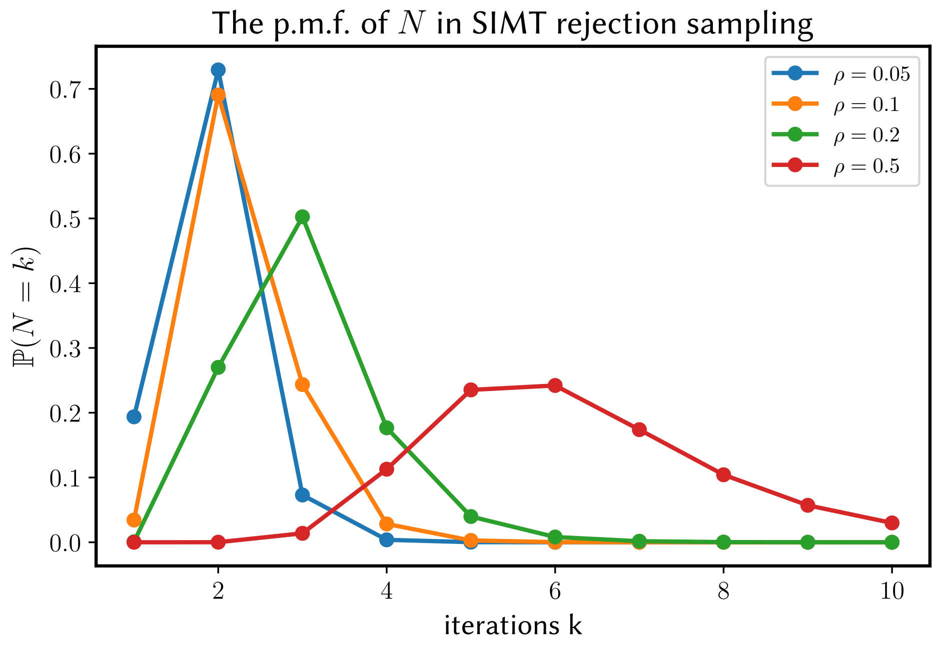
Figure: Visualization of warp divergence. The arrow indicates that the thread is doing work and the red square indicates that the thread is idle.

Analytical formula for the probability

$$P(N = k) = (1 - \rho^n)^t - \left(1 - \rho^{n-1}\right)^t,$$

where:

- N - number iterations until accept
- t - warp size (32 for NVIDIA GPUs)
- ρ - rejection probability



Methods

Implementation of Kernels

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- Flexibility in selecting kernel to benchmark without the overhead of any runtime dispatch.
- The gamma generation benchmark class use persistent threads (PT) [2], [6].

Selected Kernels

We selected 5 kernels that we believe can be be efficiently implemented on the GPU:

- Best (XG) [3]
- Cheng (GA) [4]
- Cheng-Feast (GMK3) [5]
- Ahrens-Dieter (GC) [1]
- Marsaglia-Tsang [7]

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The first four are published in the 1970s and Marsaglia-Tsang in 2000. Their measurements are on ~ 50 year old computer hardware!

Verification of implementations

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- We know mathematically that the output should be gamma distributed.

- Output quality depends on the uniform RNG. CUDAs default uniform RNG was used: `CURAND_RNG_PSEUDO_XORWOW`

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- The measurements were done for four different values of $\alpha = 1.0001, 2, 4, 10$
- Warmup iteration + 10 measurements, means are reported in figures with variance errorbars.

Experimental Setup

- Linux host running Ubuntu 22.04.3 LTS with linux kernel version 5.15.0-58-generic.
- AMD Ryzen 9 5950X 16-Core CPU with clock frequency 3.4GHz and memory listed in table 1.

L1d cache	512 KiB
L1i cache	512 KiB
L2 cache	8 MiB
L3 cache	64 MiB
RAM	32 GiB (2x16 GiB)
SSD	1TB

Table: Cache sizes for the AMD 5950X CPU used for benchmarking and installed memory sizes.

Experimental Setup

- NVIDIA GeForce RTX 4070 GPU

GPU Architecture	Ada Lovelace
CUDA Cores	5888
Clock Speed	1.92 GHz
RAM	12 GiB
Memory Interface	192-bit
Memory Bandwidth	504.2 GB/s
L1 Cache Size	192 KiB per SM
L2 Cache Size	36 MiB

Table: Key stats for the NVIDIA GeForce RTX 4070 GPU used for measurements.

Results

Verification of Output

Algorithm	$\alpha = 1.0001$		$\alpha = 2.0$		$\alpha = 10.0$	
	D_n	*p-value*	D_n	*p-value*	D_n	*p-value*
Cheng-Feast (GKM3)	0.0012	0.11	0.00094	0.34	0.0015	0.018
Marsaglia-Tsang	0.00059	0.88	0.00069	0.72	0.00072	0.67
Cheng (GA)	0.00067	0.76	0.00052	0.95	0.00074	0.64
Best (XG)	0.00069	0.73	0.00059	0.87	0.00062	0.84
Ahrens-Dieter (GC)	0.00063	0.83	0.00059	0.88	0.00064	0.80

Table: KS-test results of the algorithms for selected values of α.

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The p-values suggest that all algorithms produce gamma distributed output, except Cheng-Feast (GKM3) for high α which is much worse than the other algorithms.

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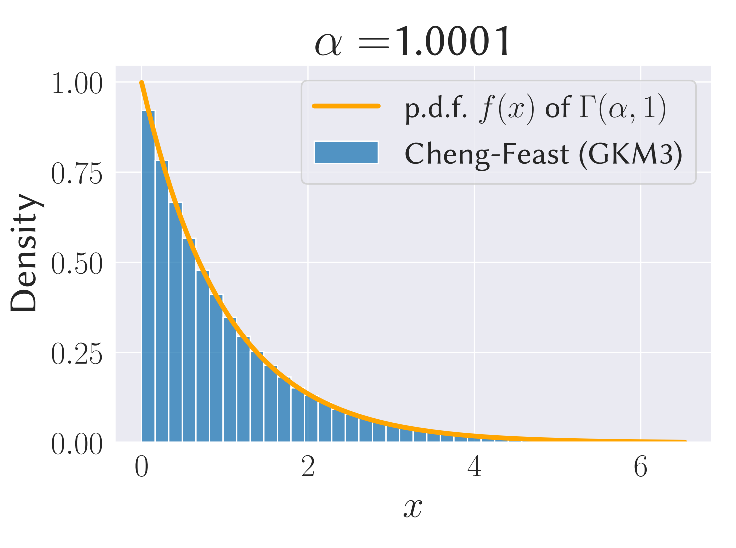
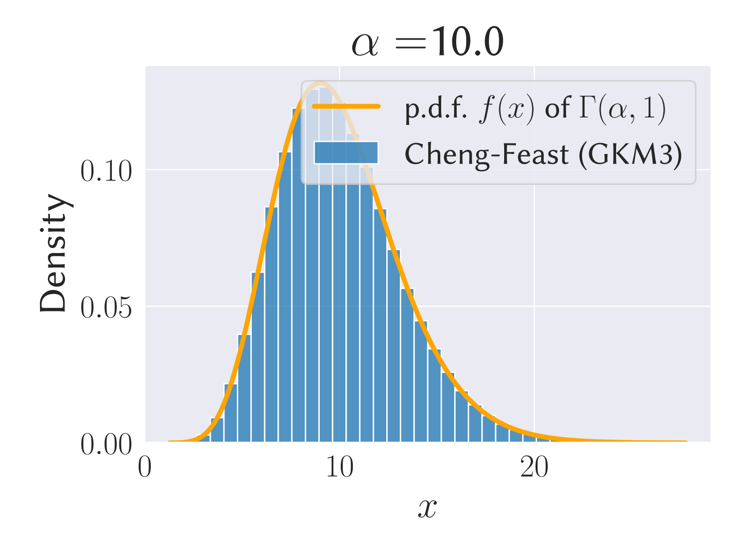


Figure: Histogram of output of Cheng-Feast (GKM3) 10^6 samples.



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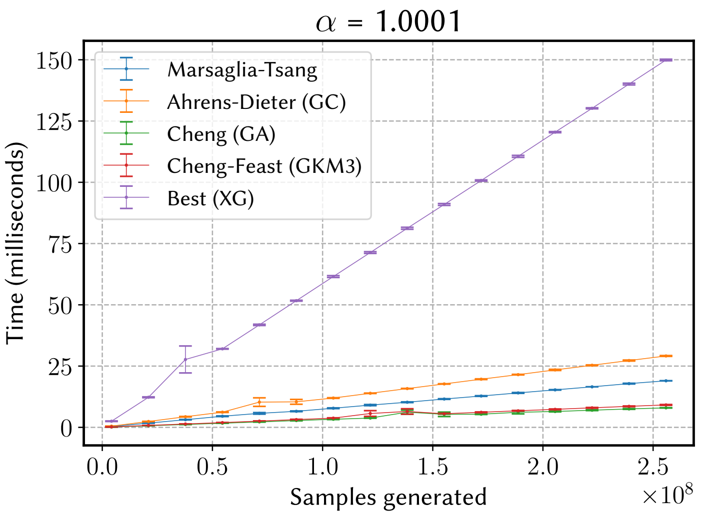


Figure: Measured execution times for α = 1.0001.

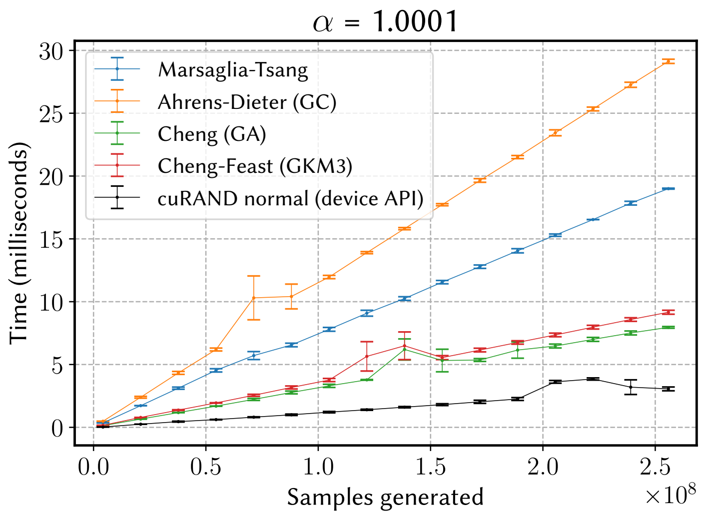
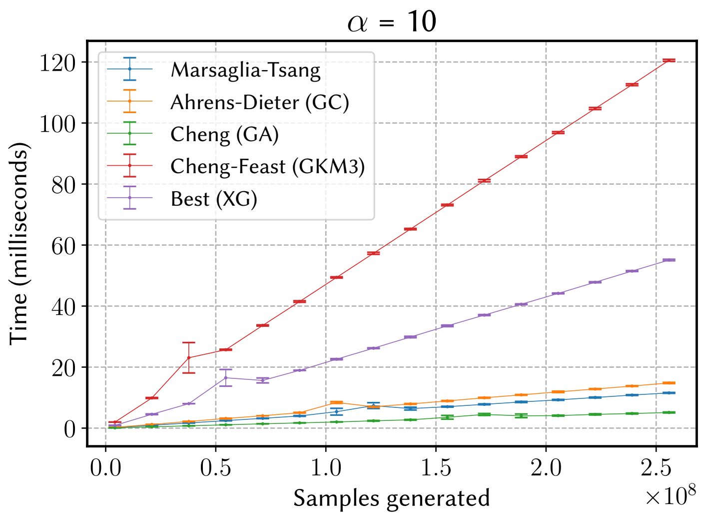


Figure: Measured execution times for the best kernels α = 1.0001 and with cuRAND normal.

Execution times α = 10



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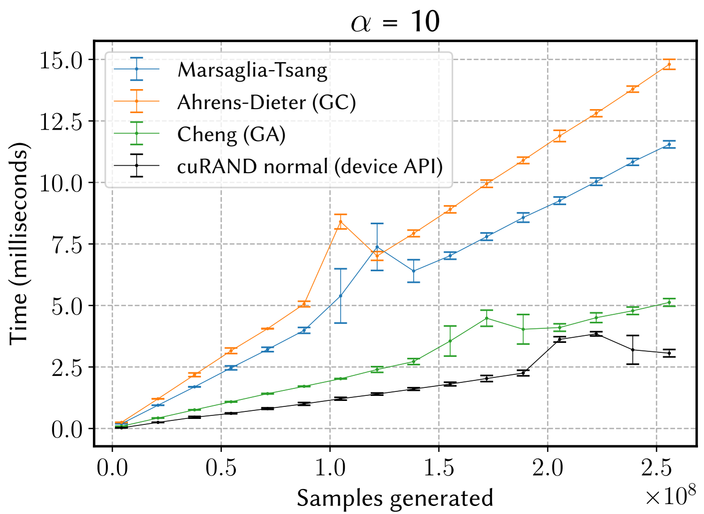

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Speedups

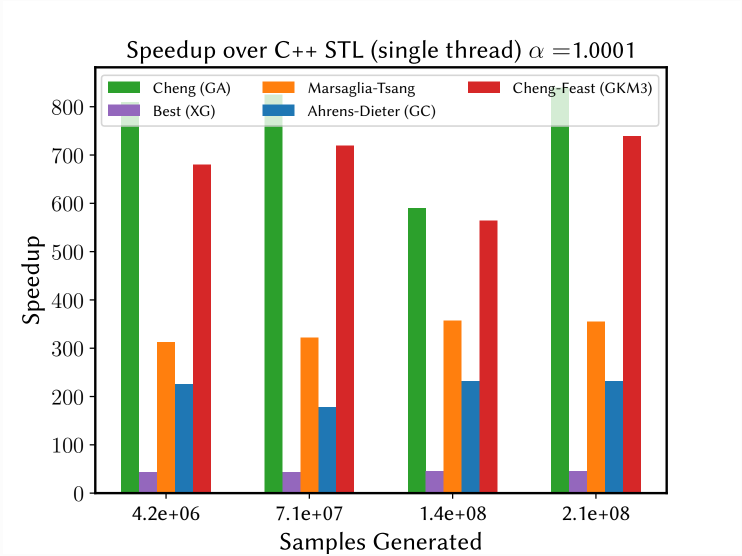
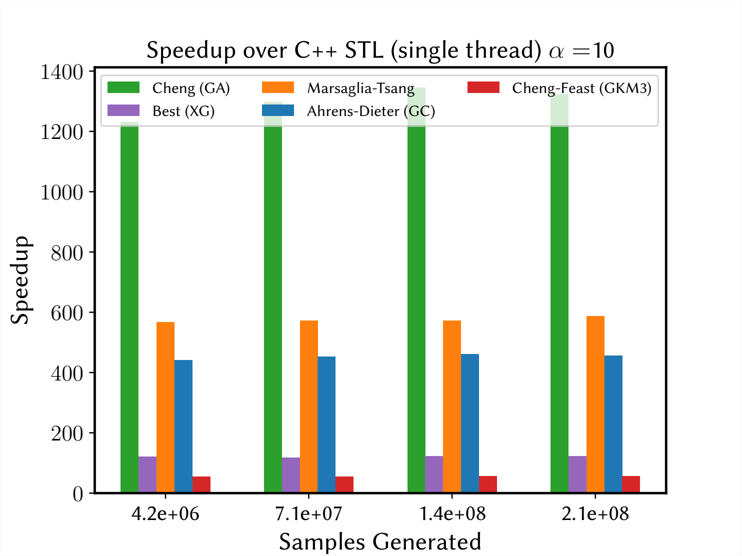


Figure: Speedup compared to CPU single thread (C++ STL) for $\alpha = 1.0001$.



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  - Achieves > $1000\times$ speedup compared to CPU for $\alpha > 2$.
  - Easy to implement ~ 25 lines of code.
- Shows that rejection sampling does not have to be ”bad” on GPUs.

Future Work

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- A natural question is whether the performance comparison on CPUs are still valid?

- The same work can be done for modern CPUs.

Questions

Feel free to ask questions!

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