Notes on Topological Dynamics with some applications in additive combinatorics

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november 2017

These notes (and the presentation) are primarily based on the lecture notes (lecture 1-4) by Terrence Tao, posted on his blog [Tao08]. It is inspired by the article *Topological dynamics and combinatorial number theory* by Furstenberg & Weiss [FW78]. Furstenberg has also written a book on on the subject and this material is covered in chapter 1 & 2 of that book [Fur].

The aim of the lecture was to introduce the basic concepts in topological dynamics needed to prove *Van der Waerden's theorem* with the approach of Furstenberg & Weiss. VW's theorem is a standard result in *Ramsey theory* [GRS90]. Which essentially is the study of how certain properties of sets are preserved under partitions. Many classical theorems in Ramsey Theory are colouring theorems of this type for example the well known *Hales-Jawett* theorem.

I feel obliged to mention that, in his lecture notes, Tao also takes the approach to the subject through ultrafilters and many of the proofs are very neat. If you are interested I recommend looking at those notes. I would also recommend looking at the book *Algebra in the Stone-Čech compactification* by Hindman & Strauss[HS98].

1 Van der Waerden's Theorem

There are two short remarks to be made about the notation used.

(1) $\mathbb{N} = \{1, 2, ...\}$ i.e. the positive integers and

(2) $\subset$ denotes proper subset, while $\subseteq$ denotes subset or equal to

Definition. A k-term **Arithmetic progression** is a sequence of natural numbers on the form: $\{u + jd\}_{j=0}^{k-1}$.
on the form:

$$\text{i.e.} \quad u, u + d, u + 2d, \ldots, u + (k - 1)d$$

Theorem 1 (Van der Warden's Theorem). *Any colouring of the positive integers into m colours will yield at least one colour which contain arbitrarily long arithmetic progressions.*

In set theoretic terms:

$$\text{Let} : \mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_m$$

then one of the sets C_j contains arbitrarily long arithmetic progressions.

There are many equivalent statements to Van der Warden's theorem, here are three of them [LR04]:

(i) $\forall\ k \geq 2$ any 2-colouring of $\mathbb{N}$ admits a monochromatic arithmetic progression of length k

The statement above was Conjectured by I.Schur and is the version VW proved in 1927.

(ii) $\forall\ k, r \geq 2$ there is a least positive integer $w(k, r)$ such that when $n \geq w$ every r-colouring of $[n]$ there is a monochromatic arithmetic progression of length k.

(iii) $\forall\ k, r \geq 2$ any r-colouring of $\mathbb{N}$ admits an arithmetic progression of length k.

2 Topological Dynamical systems

Definition. A **Topological dynamical system** (X, τ, T) is compact metrizable topological space (X, τ) with a homeomorphism $T : X \to X$.

Since X is compact and metrizable it follows from *Usrysohn's metrization theorem* that X is second-countable and Hausdorff. Connecting to C^*-algebras we can view T as a C^*-Algebra isomorphism, $T : C(X) \to C(X)$.

Definition. A subsystem (E, T) of a dynamical system (X, T) is a subset, $\emptyset \neq E \subseteq X$, which is T-invariant (i.e. $T^n E = E$) and such that $(E, T|_E)$ satisfies the properties of a dynamical system.

When considering subsystems the topology is the subspace topology. Notice that in a topological dynamical system X is Hausdorff, and since every compact subset $E \subset X$ of a Hausdorff space is closed it follows that if (E, T) is a subsystem of (X, T) then E must be closed.

Definition. A **minimal dynamical system** is a dynamical system (Y, T) with no proper subsystems. (By proper we mean a subsystem (E, T) such that $E \subset Y$.)

It follows that two minimal topological dynamical are either disjoint or conincident.

Definition. The **orbit** of a point $x \in X$ is the set

$$T^{\mathbb{Z}}(x) := \{T^n x \ : \ n \in \mathbb{Z}\}$$

It is a good guess that orbits are closely connected to minimal systems however every orbit **is not** a minimal topological dynamical system. One problem is that an orbit may not be closed and hence not compact. However for every element the closure of the orbit $\overline{T^{\mathbb{Z}} x}$ is a subsystem but this may not be a minimal system. We have the following important property which is one of the homework problems:

Problem 1. Show that (X, τ, T) is a minimal topological dynamical system **if and only if** for every element in X the orbit closure is dense in X.

A natural question to ask is whether every topological dynamical system contains a minimal dynamical system, which turns out to be true. But unfortunately every topological dynamical system can't be decomposed into a collection of minimal dynamical systems since an orbit closure may not be minimal.

Lemma 1. *Every topological dynamical system contains a minimal dynamical system.*

Proof. Let (X, τ, T) be a topological dynamical system. Consider a collection $\{E_\alpha\}$ of subsystems of X. Since X is compact and every subsystem is closed and nonempty it follows from the finite intersection property that their intersection is nonempty. Furthermore the intersection is closed and T-invariant since arbitrary intersections of closed sets are closed in a topological space and intersections of T-invariant sets are T-invariant. We can now consider this as a partially ordered chain of subsystems ordered by inclusion: $E_\alpha < E_\beta$ if $E_\beta \subset E_\alpha$. Now by Zorn's lemma there exists a maximal element which by definition is a minimal topological dynamical system. □

We classify points in a topological dynamical system after the behaviour of their orbits $T^{\mathbb{Z}}(x)$ and to do this we need to place a metric on X (with respect to our metrizable topology). The choice does not matter as the metrics are topologically equivalent. From this we conclude that it makes sense to use the metric when categorising points in the space X as their behaviour will be the same under all topologically equivalent metrics. It should also be noted that continuous functions between two compact metric spaces are uniformly continuous and it follows that the homeomorphism T is uniformly continuous. Thus:

$$\forall\, \varepsilon > 0\; \exists\, \delta > 0 \quad \text{s.t} \quad \rho(x, y) < \delta \implies \rho(T^n x, T^n y) < \varepsilon$$

Definition (Classification of points)**.** We say that:

x is **invariant** provided $T^{\mathbb{Z}}(x) = x$

x is **periodic** provided $\exists n \in \mathbb{Z}$ s.t. $T^n(x) = x$

x is **almost periodic** provided $\{n \in \mathbb{Z} \;:\; \rho(T^n(x), x) < \varepsilon\}$

is syndetic for all $\varepsilon > 0$

x is **recurrent** provided $\{n \in \mathbb{Z} \;:\; \rho(T^n(x), x) < \varepsilon\}$ is infinite for all $\varepsilon > 0$

Syndetic means that the set has bounded gaps. This means $N \subset \mathbb{Z}$ is syndetic provided there exists an integer p such that

$$\{k - p, ..., k - 1, k, k + 1, ..., k + p\} \cap N \neq \emptyset, \quad \forall\, k \in \mathbb{Z}$$

We are now ready to prove some results about recurrence in topological dynamical systems.

Theorem 2. *Let (X, τ, T) be a topological dynamical system and let $\{U_\alpha\}$ be an open cover of X. Then there exists an element of the cover, $U_{\alpha'}$ satisfying:*

$$U_{\alpha'} \cap T^n U_{\alpha'} \neq \emptyset \text{ for infinitely many } n$$

Proof. Since X is compact we can choose a finite subcover $\{U_{\alpha_n}\}$ of $\{U_\alpha\}$. Let x' be a point of X and consider the orbit:

$$T^{\mathbb{Z}} x' = \{T^n x' \ : \ n \in \mathbb{Z}\}$$

Notice that the orbit may be finite but it trivially contains x' for infinitely many n. From the infinite pigeonhole principle it follows that one of the covers say $U_{\alpha'}$ in the finite subcover must contain infinitely many n, i.e.

$$\text{there exists an infinite subset } N \subset \mathbb{Z} \text{ such that } T^N x' \subseteq U_{\alpha'}$$

The cover element will satisfy the property of the theorem. Let $n' \in N$ then by the definition of N, $T^{n'} x' \in U_{\alpha'}$. For every $n \in N$ the set $T^{-n} T^N x'$ contains x' as $T^n x' \in T^N x'$. And since $T^N x' \subset U_{\alpha'}$ it follows that:

$$T^{n'} x' \in U_{\alpha'} \cap T^{n'-n} U_{\alpha'}, \quad \forall\, n \in N$$

Which proves the theorem as N is an infinite set.

□

Lemma 2. *In a minimal topological dynamical system (X, τ, T) every element is almost periodic.*

Proof. We prove this by contradiction. Assume there exists a point $x \in X$ which is not almost periodic. i.e.

$$\exists\, \varepsilon > 0 \text{ such that } N = \{n \in \mathbb{Z} \ : \ \rho(T^n x, x) < \varepsilon\} \text{ isn't syndetic.}$$

Which means that for every $p \in \mathbb{N}$ we can find a set on the form $T_p = \{k_p - p, ..., k_p, ..., k_p + +p\}$ such that $\rho(T^k x, x) \geq \varepsilon$ for any $k \in T_p$. We can therefore construct the sequence $\{T^{k_p} x\}$ which has a convergent subsequence k_{p_j} with a limit x' in X since X is a compact metric space. Furthermore the continuity of T and the construction of the sequence k_{p_j} from the unbounded set implies that for any $n \in \mathbb{Z}$:

$$\rho(T^n x', x) = \lim_{j \to \infty} \rho(T^{k_{p_j}+n} x, x) \geq \varepsilon$$

By *Problem 1*, X is minimal **iff** the orbit of every point is dense in X. But in the above equation $\overline{T^{\mathbb{Z}}x'} \neq X$ which is a contradiction to the minimality of X. $\square$

By combining *Lemma 2.* and the fact that every topological dynamical system contains a minimal system (*Lemma 1.*) we get the following theorem:

Theorem 3 (Birkhoff recurrence theorem). *Every topological dynamical system contains a point x which is almost periodic.*

Note that this is stronger than Theorem 2 which said that we could find a an element of every open cover $U_{\alpha'}$ such that $U_{\alpha'} \cap T^n U_{\alpha'} \neq \emptyset$ for infinitely many n. By choosing a cover element, U_{α^*} which contains an almost periodic point we can even say that the set $N = \{n \in \mathbb{Z} \ : \ T^n U_{\alpha^*} \cap U_{\alpha^*} \neq \emptyset\}$ is syndetic.

The following theorem is The topological counterpart of Van der Waerden's Theorem.
Theorem 4 (Topological Van der Waerden's Theorem). *Let $\mathcal{C} = \{U_\alpha\}$ be an open cover of Topological dynamical system (X, τ, T) and $k \in \mathbb{N}$ then there exists a set, $U_{\alpha'}$, in $\mathcal{C}$, satisfying:*

$$U_{\alpha'} \cap T^{-r} U_{\alpha'} \cap \cdots \cap T^{-(k-1)r} U_{\alpha'} \neq \emptyset \textit{ for infinitely many } r$$

The idea of how ones goes from Theorem 4 to the standard combinatorial Van der Waerden's Theorem is by considering the power set, $\Omega = \Gamma^{\mathbb{Z}}$, where $\Gamma = \{1, 2, ..., r\}$. Thus one can think of an element ω of Ω as a function $\omega : \mathbb{Z} \to \Gamma$. In fact Ω is a compact metric space under the metric

$$\rho(\omega_1, \omega_2) := \inf \left\{ \frac{1}{m+1} \ : \ \omega_1(n) = \omega_2(n) \text{ for } |n| < m \right\}$$

and the shift map $S : \Omega \to \Omega$ by $\omega(n) \mapsto \omega(n+1)$ is a homeomorphism. Then Ω contain the colouring map, c, corresponding to our partition of $\mathbb{N}$ defined by:

$$\begin{cases} c(n) = m \text{ if } n \in C_m \text{ \& if } n > 0 \\ c(n) = 1, \text{ if } n \leq 0 \end{cases}$$

If you want to read the full proof I recommend the original article by Furstenberg & Weiss [FW78]

More generally given a compact metrizable space Γ one can always define a metric on the space $\Gamma^{\mathbb{Z}}$ by

$$\rho\left(\{\gamma_n^1\}_{n\in\mathbb{Z}},\{\gamma_n^2\}_{n\in\mathbb{Z}}\right):=\sum_{n\in\mathbb{Z}}2^{-|n|}\rho_\Gamma(\gamma_n^1,\gamma_n^2)$$

Where $\gamma^i\in\Gamma^{\mathbb{Z}}$ is represented by the sequence, $\{\gamma_n^i\}_{n\in\mathbb{Z}}$ in Γ.

The proof of *Theorem 4.* goes by *Theorem 5.*

Theorem 5. *Let $\mathcal{C}=\{U_\alpha\}$ be an open cover of Topological dynamical system (X,τ,T) and $k\in\mathbb{N}$ then there exists a set, $U_{\alpha'}$, in $\mathcal{C}$, which contains an arithmetic progression:*

$$\{T^r x,T^{2r}x,\ldots,T^{(k-1)r}x\}\subseteq U_{\alpha'}\ \textit{for some}\ x\in X\ \textit{and}\ r\in\mathbb{N}$$

Theorem 5. actually implies *Theorem 4.* This can be proven by considering the system $(X\times\mathbb{Z}_n,S)$ with the map $S:(x,m)\mapsto(Tx,m+1)$. For a fixed k, consider the open cover $\{U_\alpha\times\{m\}\}$ with α in some index set A and $m\in\mathbb{Z}_n$. By *Theorem 5.* we know that there exists a cover element $U_{\alpha'}\times\{j\}$ which contains an arithmetic progression of the form $(x,m),(Tx,m+1),...,(T^{(k-1)r}x,m+(k-1)r)$. But there will also be an open cover if we increase n to say N and instead consider the system $(X\times\mathbb{Z}_N,S)$. But if we choose N large enough we can make sure that that the set $\{j\}$ does not contain $\{r,...,(k-1)r\ (mod\ N)\}$ for any $j\in\{1,...,N\}$. However there will still exist a recurrent set as in *Theorem 5.* but for a larger value on r. By this method one can show that for any $r>0$ there is a cover element, $U_{\alpha'}$ which contains a set $\{x,Tx,T^r x,...,T^{(k-1)r}x\}$. And to show that there exists a cover element such that this holds for infinitely many r, for a given k, one simply covers X by a finite subcover and applies the infinite pigeonhole principle.

In order to prove *Theorem 5.* we must use the following *Lemma.*

Lemma 3. *Let (X,τ,T) be a minimal topological dynamical system and $U\subseteq X$ be nonempty and open. Then X can be covered by a finite number of translates of U, i.e.*

$$X=\bigcup_{j=1}^{k}T^{n_j}U$$

Proof. Since T is a homeomorphism and U is open, every translate T^nU will also be open. Therefore

$$\bigcup_{n\in\mathbb{Z}}T^nU \text{ is open which implies that the set } X\setminus\bigcup_{n\in\mathbb{Z}}T^nU \text{ is closed}$$

However since X is minimal this closed set can't be a subsystem and must therefore be the empty set. Which means that the collection $\{T^n U\}_{n\in\mathbb{Z}}$ is an open cover of X. Compactness implies that there is a finite subcover.

$$\therefore \quad \exists \, \{n_j\}_{j=1}^{k} \subset \mathbb{Z} \text{ such that } X = \bigcup_{j=1}^{k} T^{n_j} U$$

□

The next theorem follows directly from *Lemma 3.* and *Theorem 5.*

Theorem 6. *Let (X, τ, T) be a minimal topological dynamical system and $U \subseteq X$ be nonempty and open. Then for every $k \in \mathbb{N}$ there is an arithmetic progression:*

$$\{T^r x, T^{2r} x, \ldots, T^{(k-1)r} x\} \subseteq U \textit{ for some } x \in X \textit{ and } r \in \mathbb{N}$$

We are now ready to prove *Theorem 5.* It is a proof by induction over k and the core idea is contained in the *colour focusing Lemma.*

Proof of Theorem 5. First consider the case $k = 1$. Then the *Theorem 5.* follows from *Theorem 2.*

Next assume that *Theorem 5.* is true for $k - 1$. Let (Y, τ, T) be a minimal dynamical subsystem of X. And let $\{U_\alpha\}$ be a finite open cover of Y. In order to show that one of the cover elements $U_{\alpha'}$ contains an arithmetic progression of the form $\{x, T^r x, \ldots, T^{(k-1)r} x\}$ we use the following lemma

Lemma 4 (Colour focusing)**.** *Let (Y, τ, T) be a minimal dynamical system and $\{U_\alpha\}$ an open cover. Then for any $j \geq 0$ there is a sequence $x_0, \ldots, x_j$ of points in X, a sequence, $U_{\alpha_0}, \ldots, U_{\alpha_j}$, of sets in the open cover (may not be distinct), and a sequence $r_1, \ldots, r_j$ of positive integers satisfying:*

$$T^{i(r_{a+1}+\cdots+r_b)} x_b \in U_{\alpha_a} \textit{when} \begin{cases} 0 \leq a \leq b \leq j \\ 1 \leq i \leq k-1 \end{cases}$$

Proof. This is proved by induction over j. The case when $j = 0$ is simply the trivial case $T^0 = Id$ and is obviously true.

Next assume that the lemma holds for $j - 1$. Then there exists sequences $r_1, ..., r_{j-1}$, $U_0, ..., U_{j-1}$, and $x_0, ..., x_{j-1}$ such that

$$T^{i(r_{a+1}+\cdots+r_b)} x_b \in U_{\alpha_a}, \quad \textit{when} \begin{cases} 0 \leq a \leq b \leq j-1 \\ 1 \leq i \leq k-1 \end{cases}$$

Theorem 5. holds for $k-1$ and therefore *Theorem 6.* also holds for $k-1$. Consider an open set V containing x_{j-1}. By *Theorem 6.* there is a subset $\{y, T^r y, ..., T^{(k-2)r} y\}$. Let $r_j = r$ and define $x_j := T^{-r_j} y$. Furthermore choose a cover element U_j containing x_j then provided V is as above and contained in U_j (we can always find such a V) we get

$$T^{i(r_{a+1}+\cdots+r_j)} x_j = T^{i(r_{a+1}+\cdots+r_{j-1})} T^{(i-1)r_j} y \in T^{i(r_{a+1}+\cdots+r_{j-1})} V$$

Notice that $i-1 < k-1$ hence $T^{(k-1)r_j} y \in V$. And this holds for $1 \leq a < j$ and $i \leq i \leq k-1$ which proves the Lemma. $\square$

To complete the proof of *Theorem 5.* we consider the finite cover of Y. Let j be the cardinality of the finite cover of Y, i.e. $|\{U_\alpha\}| = j$. Then by the lemma for j there must be two cover elements in the collection of $j+1$ (not necessarily distinct) cover elements which are equal, say $U_{\alpha_a} = U_{\alpha_b}$. Define $x := x_b$ and $r := r_{a+1} + \cdots + r_b$ then it follows that

$$\{x, Tx, \ldots, T^{(k-1)r}\} \subset U_{\alpha_a} = U_{\alpha_b}$$

$\square$

A generalisation of the Topological Van der Waerden's theorem is the *Multiple Birkhoff Recurrence Theorem* and this can be applied to prove equidistributional properties of sequences and that's used in Problem 2.

Theorem 7 (Multiple Birkhoff recurrence)**.** *For any $k \in \mathbb{N}$ every topological dynamical system (X, τ, T) contains a point $x \in X$ s.t. there is a sequence of integers $\{r_j\} \to \infty$ satisfying:*

$$\lim_{j\to\infty} T^{ir_j} x = x, \textit{ for all } 0 \leq i \leq k-1$$

Problem 2. Use the Multiple Birkhoff Recurrence theorem to show that for any real number α there is a sequence of a integers $n_j \to \infty$, such that:

$$\lim_{j\to\infty} \operatorname{dist}(n_j^2 \alpha, \mathbb{Z}) = 0$$

Hint: Consider $X = (\mathbb{R}/\mathbb{Z})^2$ and the map $T(x, y) = (x + \alpha, x + y)$.
Remark: You must show that the system is indeed a TDS

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