# Complex Analysis

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My intention when writing this text was to summarize the most basic concepts and central theorems of Complex Analysis in a concise manner. Hopefully you will find it a useful reference. If you find any errors please email me at:

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# Complex Identities

Some notation:
Let $z, w \in \mathbb{C}$. If $z = \alpha + i\beta$ then $\overline{z} := \alpha - i\beta$ denotes the **conjugate** of $z$.

### Absolute value

$|z| = \sqrt{\alpha^2 + \beta^2}$

$|z + w| \leq |z| + |w| \quad$ (*triangle inequality*)

$|z - w| \leq |z| - |w| \quad$ (*reverse triangle inequality*)

$|zw| = |z||w|$

$|z|^2 = z\overline{z}$

$\frac{1}{z} = \frac{\overline{z}}{|z|^2} \quad$ (when $z \neq 0$)

$Re(z) \leq z, \quad Im(z) \leq z$

$Re(z) = \frac{z + \overline{z}}{2}, \quad Im(z) = \frac{z - \overline{z}}{2i}$

### Conjugate

$\overline{z + w} = \overline{z} + \overline{w}$

$\overline{zw} = \overline{z}\overline{w}$

### Trignometric identities

$z = re^{i\theta}$

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$

$\cosh(z) := \frac{e^{z} + e^{-z}}{2} \quad \sinh(z) := \frac{e^{z} - e^{-z}}{2}$

**The logarithm function**

$\arg z := \theta, \quad \text{when } z = re^{i\theta} \implies arg(z) = \varphi + 2k\pi \quad \varphi \in (-\pi, \pi], \quad k \in \mathbb{Z}$

$\operatorname{Arg} z$ is said to be **the principal value** of $\arg z$ and is defined by

$$\operatorname{Arg} z := \varphi \in (-\pi, \pi] \text{ when } \arg z = \varphi + 2\pi k \quad k \in \mathbb{Z}$$

We define the logarithm for $z \in \mathbb{C} \setminus \{0\}$ as

$$\log z := \operatorname{Log}|z| + i \arg z \quad (\operatorname{Log}|z| \text{ is the natural logarithm (i.e. } \ln|z|)$$

The **principal value of the logarithm** is defined as

$$\operatorname{Log} z := \operatorname{Log}|z| + i \operatorname{Arg} z$$

$\operatorname{Log} z$ is analytic in the Domain $\Omega = \mathbb{C} \setminus (-\infty, 0]$ with derivative

$$\frac{d}{dz} \operatorname{Log} z = \frac{1}{z} \quad \text{for } z \text{ in } \Omega$$

**Complex Powers**

The definition of a complex power for $z \neq 0$ and $\alpha \in \mathbb{C}$ is given by

$$z^\alpha := e^{\alpha \log z}$$

The **principal branch of** $z^\alpha$ is given by $e^{\alpha \operatorname{Log} z}$. This function is analytic in the domain $\Omega$ defined above with the derivative

$$\frac{d}{dz} e^{\alpha \operatorname{Log} z} = \{\text{the chain rule}\} = \frac{\alpha}{z} e^{\alpha \operatorname{Log} z}$$

**Harmonic functions**

A function $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic if

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

If a function $f(z) = u(x, y) + iv(x, y)$ is holomorphic in an open set $\Omega$ then the functions $Re(z) = u$ and $Im(z) = v$ are harmonic.

# Convergence and Completeness of $\mathbb{C}$

A sequence $\{z_n\}$ in $\mathbb{C}$ is said to converge to $w \in \mathbb{C}$ if:

$$\lim_{n\to\infty} |z_n - w| = 0, \quad \text{we write} \quad \lim_{n\to\infty} z_n = w \quad \textit{or} \quad z_n \to w \text{ as } n \to \infty$$

**Lemma:** $z_n \to w$ *if and only if* $\begin{cases} \lim_{n\to\infty} Re(z_n) = Re(w) \\ \lim_{n\to\infty} Im(z_n) = Im(w) \end{cases}$

**Definition** A sequence is said to be a *Cauchy seqeunce* or simply *Cauchy* if:

given $\varepsilon > 0 \quad \exists N \in \mathbb{N} \quad : \quad n, m \geq N \implies |z_n - z_m| < \varepsilon$

**Remark.** $\{z_n\}$ *is Cauchy if and only if* $\{Re(z_n)\}$ and $\{Im(z_n)\}$ *are.*

**Theorem:** $\mathbb{C}$ *is complete in the sense that every* **Cauchy sequence** *in* $\mathbb{C}$ *converges.*

# Sets in the complex plane and some topology

Let $z_0 \in \mathbb{C}$ and $r \in \mathbb{R}_+$

$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$ *The open disc centered at $z_0$ with radius $r_0$*

$\overline{D}_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\}$ *The closed disc centered at $z_0$ with radius $r_0$*

$C_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| = r\}$ *The circle centered at $z_0$ with radius $r_0$*

$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ *The unit disc*

**Definition:** Let $\Omega \subset \mathbb{C}$. $z_0$ is an *interior point* of $\Omega$ if:

$$\exists r > 0 \quad : \quad D_r(z_0) \subset \Omega$$

**Definition:** A set $\Omega \subset \mathbb{C}$ is *open* if every point of the set is an interior point.

**Definition:** $\Omega$ is closed if $\Omega^{\complement} = \mathbb{C} \setminus \Omega$ is open.

**Lemma:** *A set is closed if and only if it contains all its limit points*

**Definition:** A point $z \in \Omega$ is called a *limit point* of $\Omega$ if there exists a sequence:

$$\{z_n\} \in \Omega \quad : \quad z_n \to z$$

**Equivalently** $z$ is a limit point of $\Omega$ if every open neighbourhood of $z$ intersects $\Omega$ in some point other than $z$.

**Definition:** The *closure* of $\Omega$, denoted $\overline{\Omega}$ is the intersection of all closed sets containing $\Omega$. It follows that:

$$\overline{\Omega} = \Omega \cup \{\text{limit points of } \Omega\}$$

**Definition:** A collection of open sets $\mathcal{C}$ such that

$$\Omega \subseteq \bigcup_{U \in \mathcal{C}} U$$

is called an *open cover* of $\Omega$

**Definition:** A set $\Omega$ is *compact* if every open cover of

Since the complex numbers are a comlete metric space the classical *Heine-Borel Theorem* holds:

**Theorem:** *(Heine-Borel) Every closed and bounded subset of* $\mathbb{C}$ *is compact.*

**Definition:** A set $\Omega$ is bounded if there exists number:

$$M \in \mathbb{R}_+ \quad : \quad z \in \Omega \implies |z| \leq M$$

**Remark.** $\Omega \subset \mathbb{C}$ is compact if and only if every sequence in $\Omega$ has a subsequence that converges to a point in $\Omega$.

**Definition:** The *diameter* of $\Omega$;

$$\text{diam}(\Omega) := \sup_{z,w \in \Omega} |z - w|$$

**Lemma:** If $\Omega_1 \supset \Omega_2 \supset ... \supset \Omega_n \supset ...$ is a sequence of non empty compact sets in $\mathbb{C}$ with the property:

$$\text{diam}(\Omega_n) \to 0 \quad \text{as} \quad n \to \infty$$

then there exists a unique $w \in \mathbb{C}$ such that $w \in \Omega_n$ for all $n \in \mathbb{N}$

# Homotopy and simply connected domains

From now on when talking about a **curve** what is really meant is a **piecewise smooth curve**. Let $\gamma_0, \gamma_1$ be two closed curves in a topological space $X$, with parameter interval $I = [0, 1]$. The curves $\gamma_0, \gamma_1$ are said to be **homotopic** if there is a continuous mapping

$$H : I^2 \to X \quad : \quad H(s,0) = \gamma_0(s), \quad H(s,1) = \gamma_1(s), \quad H(0,t) = H(1,t)$$

If two curves are homotopic this means that they can be continuously deformable into each other within the space $X$.

A curve $\gamma$ is said to be **null-homotopic** in $X$ if it is *homotopic to a constant mapping* (i.e. a point in $X$).

A **simply connected domain** in $\mathbb{C}$ is a connected subset $\Omega \subset \mathbb{C}$ in which *every closed curve is null-homotopic*

This means that **every closed curve, $\gamma_1$, in a simply connected domain $\Omega$ is homotopic to and can be continuosly deformed into any other closed curve $\gamma_2$ in $\Omega$.** Some important results follow from this which will not be proven here as they would require involving new concepts. Proofs of some of the following statements and theorems can be done by introducing the concept called *index* or *winding number* of a point with respect to the curve[1]. A fair amount of topology is also needed[2].

**Theorem** *(Deforming invariance theorem)*
*Let $\Omega$ be a domain (open connected subset of $\mathbb{C}$). If the closed curves $\gamma_0, \gamma_1$ are homotopic in $\Omega$ and $f \in H(\Omega)$ then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

**It follows that in a simply connected domain the value of an integral of a function holomorphic in that domain is the same for any closed curve in that domain.** Therefore when proving results, such as Cauchy's Theorem for a circle, subset of a simply connected domain, the same result will hold for any closed curve in the domain.

[1] More information on the subject may be found in Rudin's *Real and Complex Analysis* or Ahlfors *Complex Analysis*

[2] Which can be found ind Munkres' *Topology* or Lee's *Introduction to Topological Manifolds*

# Important Theorems and Results

**Theorem** *(Cauchy Riemman Equations)*
*Let $f \in H(\Omega)$ then f can be expressed as $f = u(x, y)+iv(x, y)$ and furthermore $u, v$ satsify the Cauchy Riemann Equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y}$$

**Proof idea** Since $f$ is holomorphic in $z$ then the limit in the derivative must be equal if we approach z from the real axis and the imaginary axis. Compare those limits and this Implies that Cuachy Riemann are satisfied.

Notice that the following proof of Cauchy's Integral theorem does not rely on the continuity of the partial derivatives $u(x, y), v(x, y)$ as the proof taking the vector analysis approach with Green's theorem.

**Cauchy's Integral Theorem for a disc** *If f is holomorphic in a disc, then for any closed curve $\gamma \subset D$*

$$\int_\gamma f(z)dz = 0$$

The idea of the proof is to show that f has a primitive inside the disc and then the integral is easily calculated and 0 since the curve is closed.

**Step 1.** *Lemma (Goursat's Theorem)*
*Let $T$ be a closed triangle and $T \subset \Omega$ open. The for any $f \in H(\Omega)$*

$$\int_{\partial T} f(z)dz = 0$$

It follows that if $R$ is a closed rectangle s.t. $R \subset \Omega$ then

$$\int_{\partial R} f(z)dz = 0$$

**Step 2.** Show that a holomorphic function in a disc $D$ has a primitive in that disc.

Without loss of generality assume the disc is centered at 0. Since this can be achieved by translation. Define the curve $\gamma_z$ as the horizontal line segment from 0 to $Re(z)$ and then the vertical line segment from $Re(z)$ to $z$. Let

$$F(z) = \int_{\gamma_z} f(w)dw$$

We will show that this is the primitive of $f(z)$ in the disc. Consider

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)dw - \int_{\gamma_z} f(w)dw$$

For small $h$ so that $(z+h) \in D$

With some geometry and Goursat's Theorem it follows that

$$F(z+h) - F(z) = \int_{\nu} f(w)dw$$

Where $\nu$ is the straight curve joining $z$ and $z+h$. Since $h$ is small and $f$ is continuous we can express

$$f(w) = f(z) + \psi(w) \quad \text{where } \psi(w) \to 0 \text{ as } w \to z$$

Hence

$$F(z+h) - F(z) = \int_{\nu} f(z)dw + \int_{\nu} \psi(w)dw = f(z)h + \int_{\nu} \psi(w)dw$$

For the second term we have

$$\lim_{h\to 0} \left| \int_{\nu} \psi(w)dw \right| \leq \lim_{h\to 0} h \sup_{w\in\nu} |\psi(w)| = 0 \implies \lim_{h\to 0} \int_{\nu} \psi(w)dw = 0$$

Thus

$$\lim_{h\to 0} F(z+h) - F(z) = \lim_{h\to 0} f(z)h \iff F'(z) = f(z)$$

**Step 3.** Since $f$ has a primitive in $\Omega$ the integral is easily calculated

$$\int_{\gamma} f(z)dz = F(z_0) - F(z_0) = 0$$

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**Cauchy's Integral formula** let $f \in H(\Omega)$ and $\overline{D} \in \Omega$. If $C$ denotes the boundary of $D$ in positive orientation, then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw \quad \textit{for } z \in D$$

The main idea of the proof is to consider the keyhole contour and letting the radius of its circle and the width approach zero.

Let $\Gamma_{\delta,\varepsilon}$ be the keyhole which inner circle is centered at z and has radius $\varepsilon$ with the width of the corridor $\delta$. By Cauchy's Integral theorem we know that the integral over $\Gamma_{\delta,\varepsilon}$ is 0. And $\gamma_1 \to -\gamma_2$ when $\delta \to 0$. Hence

$$\lim_{\delta \to 0} \int_{\Gamma_{\delta,\varepsilon}} \frac{f(w)}{w-z} dw = 0 = \int_C \frac{f(w)}{w-z} dw + \int_{C_\varepsilon} \frac{f(w)}{w-z} dw$$

Equivalently:

$$\int_C \frac{f(w)}{w-z} dw = -\int_{C_\varepsilon} \frac{f(w)}{w-z} dw$$

Now rewrite

$$\frac{f(w)}{w-z} = \frac{f(w)-f(z)}{w-z} + \frac{f(z)}{w-z}$$

thus

$$\int_{C_\varepsilon} \frac{f(w)}{w-z} dw = \int_{C_\varepsilon} \frac{f(w)-f(z)}{w-z} dw + \int_{C_\varepsilon} \frac{f(z)}{w-z} dw$$

Considering the first term

$$\int_{C_\varepsilon} \frac{f(w)-f(z)}{w-z} dw \leq l(C_\varepsilon) \sup_{w \in C_\varepsilon} \left| \frac{f(w)-f(z)}{w-z} \right| \leq 2\pi\varepsilon \sup_{w \in C_\varepsilon} \frac{|f(w)-f(z)|}{\varepsilon}$$

Letting $\varepsilon \to 0$ yields the following result as $f$ is continuous

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} \frac{f(w)-f(z)}{w-z} dw \leq 2\pi |f(w) - f(w+\varepsilon)| = 0$$

We get

$$\lim_{\varepsilon \to 0} \int_C \frac{f(w)}{w-z} dw = \int_C \frac{f(w)}{w-z} dw = \lim_{\varepsilon \to 0} - \int_{C_\varepsilon} \frac{f(z)}{w-z} dw = -f(z) \int_{C_\varepsilon} \frac{dw}{w-z}$$

Since $C_\varepsilon$ is centered at $z$ with clockwise direction we can parametrize the curve as

$$\begin{cases} w = z + \varepsilon e^{-i\varphi} \\ dw = -\varepsilon i e^{-i\varphi} d\varphi \end{cases} \quad \text{for } \varphi \in [0, 2\pi]$$

With this parametrization it is clear that

$$\int_C \frac{f(w)}{w-z} dw = -f(z) \int_0^{2\pi} i d\varphi = f(z) 2\pi i \iff f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}$$

■

**Remarks**

(i) This holds for any contour Γ homotopic to C inside the simply connected domain Ω in which f is holomorphic.

(ii) If $z$ is a point outside of Γ then the integral vanishes. (follows from Cauchy's Integral Theorem)

(iii) By induction one can show that under the assumptions above, $f$ has *infinitely many complex derivatives in* Ω and those are given by

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^{n+1}}dw \quad \text{for } z \text{ in the interior of } \Gamma$$

**Lemma** *(Cauchy's Inequalities)*
*Let $\Omega \subset \mathbb{C}$ be open such that $f \in H(\Omega)$ and $\overline{D_R}(z_0) \subset \Omega$. If $C$ denotes the boundary of $D_R(z_0)$ then*

$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n}\sup_{z\in C}|f(z)|$$

**Proof.** Simply apply Cauchy's Integral formula

$$|f^{(n)}(z_0)| = \left|\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| = \frac{n!}{2\pi}\left|\int_0^{2\pi}\frac{f(z_0+Re^{i\varphi})}{(Re^{i\varphi})^{n+1}}d\varphi\right| \leq \frac{n!}{2\pi}\frac{1}{R^n}l([0,2\pi])\sup_{z\in C}|f(z)|$$

**Theorem** *(Liouville's Theorem) If f is entire and bounded then f is constant*

**Theorem** *(Fundamental Theorem of Algebra)*
*Every nonconstant polynomial with complex coefficients has at least one complex root*

**Theorem** *On convergence of the taylor series of holomorphic functions*
*Let f be holomorphic in $D_R(z_0)$ then the Taylor series for f around $z_0$ converges for every $z \in D_R(z_0)$.* This theorem implies that Taylor series will converge to f(z) everywhere inside the largest open disk centered at $z_0$, over which f is holomorphic.

**Theorem** *On the convergence of the Laurent series Let f be holomorphic in an annulus or punctured disc $D^* = \{z \in \mathbb{C} : r < |z-z_0| < R\}$ then f can be expressed there as the sum of two series which converges in to f in $D^*$*

$$f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j + \sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j} = \sum_{j=-\infty}^{\infty} a_j(z-z_0)^j$$

whose coefficients are given by

$$a_j = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw$$

for any closed curve $C$ in $D^*$ positively oriented around $z_0$

## Meromorphic functions and Residuals

An **isolated singularity** of a complex function is a point $z_0$ such that $f$ can be defined in some neighbourhood of the point but not at the point itself. There are three types

1. $z_0$ is a **removable** singularity *iff* $|f|$ is bounded near $z_0$ *iff* f has a limit as $z \to z_0$ *iff* f can be redefined so that it is holomorphic at $z_0$
2. $z_0$ is a **pole** *iff* $|f(z)| \to \infty$ as $z \to z_0$ *iff* $f$ can be written $\frac{g(z)}{(z-z_0)^n}$ for some $n \in \mathbb{N}$ and holomorphic function $g : g(z_0) \neq 0$
3. A singularity which is neither of the above is called an **essential singularity**

**Theorem** *(Picard's Theorem) A function with an essential singularity assumes every complex number, with possibly one exception, as a vlue in any open neighbourhood of this singularity.*

**Theorem** *If $f$ has a pole of order $n$ at $z_0$ then*

$$\text{Res}[f; z_0] = \lim_{z\to z_0} \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}(z-z_0)^n f(z)$$

**Residual formula** *If $f$ is holomorphic in an open set containing contour $\Gamma$ with positive orientation and its interior, except for a finite number of poles $z_0, ..., z_n$ inside $\Gamma$ then*

$$\int_\Gamma f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}[f; z_k]$$

**Jordan's Lemma** *If $m > 0$ and $P, Q$ are two polynomials such that*

$$deg(Q) \geq 1 + deg(P)$$

then

$$\lim_{\rho \to \infty} \int_{C_\rho^+} e^{imz} \frac{P(z)}{Q(z)} dz = 0$$

Where $C_\rho^+$ denotes the upper half-circle of radius $\rho$.

**Definition** A function $f$ on open set $\Omega$ is **meromorphic** if there is a set $A \subset \Omega$ such that

1. $A$ has no limit points in $\Omega$
2. $f$ has poles at every point of $A$
3. $f$ is holomorphic in $\Omega \setminus A$

**Remark** This implies that every holomorphic function is meromorphic as well.

**Theorem** *(Argument principle)*
*If $f$ is holomorphic on the positively oriented closed curve $\Gamma$ and merormorphic in its interior then*

$$N_0(f) - N_p(f) = \frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} dz$$

Where $N_0(f)$ denotes the number of zeros of $f$ inside $\Gamma$ and $N_p(f)$ denotes the number of poles of $f$ inside $\Gamma$. (Multiplicities of zeros and poles included)

**Corollary** *If $f$ is holomorphic on and inside the positively oriented closed curve $\Gamma$ then*

$$N_0(f) = \frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} dz$$

**Theorem** *(Rouché's Theorem)*
*If $f$ and $g$ are holomorphic on and inside the positively oriented closed curve $\Gamma$ and if*

$$|h(z)| \leq |f(z)| \quad \text{on } \Gamma$$

Then

$$N_0(f) = N_0(f + h) \quad \text{inside } \Gamma$$

**Theorem** *(Open mapping theorem) If f is holomorphic and non-constant in a domain (connected open set) $\Omega$ then f is open. (i.e. maps open sets to open sets)*

**Theorem** *(Maximum modulus principle)* If f is a non-constant holomorphic function in a domain $\Omega$ then f cannot attain a maximum in $\Omega$.

# Conformal Mappings

A **linear fractional transformation** commononly denoted Möbius transformation is a mapping in the form

$$w = S(z) = \frac{az + b}{cz + d}$$

With $ad \neq bc$. And has the inverse

$$z = S^{-1}(w) = \frac{dw - b}{-cw + a}$$

**Theorem** *Let S be a möbius transformation then*

1. $S$ maps the class of circles and lines to itself. If a circle or a line passes through the pole $z = -d/c$ of $S$ then it gets mapped to as straight line. Any line or circle that avoids the pole of $S$ will be mapped to a circle.
2. $S$ maps the extended complex plane innjectively onto itself.

A conformal mapping maps the left region of a curve to the left region of its picture.

**The cross ratio** Given three points $z_2, z_3, z_4$ in the extended complex plane there is a linear transformation S which carries the points into $1, 0, \infty$ in this order. If none of the points is $\infty$ then

$$S(z) = \frac{z - z_3}{z - z_4}\frac{z_2 - z_3}{z_2 - z_4}$$

If one of the points $z_2, z_3, z_4$ is $\infty$ (only one can be) then in order

$$S(z) = \frac{z - z_3}{z - z_4}, \quad S(z) = \frac{z_2 - z_4}{z - z_4}, \quad S(z) = \frac{z - z_3}{z_2 - z_3}$$

The image of $z_1$ under the above transformation is called the **cross ratio**, denoted $(z_1, z_2, z_3, z_4)$

# Appendix

**Lemma:** *If* $f \in H(\Omega)$ *has a Primitive* $F$ *and* $\gamma$ *is a piecewise smooth curve in* $\Omega$ *with endpoints* $w_1, w_2$ *then:*

$$\int_\gamma f(z)dz = F(w_2) - F(w_1)$$

**ADD PROOF OF GOURSAT**